The asymptotic shape of the branching random walk

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1 Introduction

Notations.

Let's think about branching random walk on $\mathbb{R}^n$. An initial ancestor starts at the origin.

$\{Z_{r_1}^{(1)}\} = \{Z_r^{(1)}\}$: a set of positions of the first generation people. All initial ancestors can make first generation people in positions in $\{Z_r^{(1)}\}$. Assume that And assume that the expected number of people in the first generation is strictly greater than 1.

$\{Z_{r_n}^{(n)}\}$: set of positions of the people in n-th generation.

$\mathfrak{F}^n$: $\sigma-$field generated by all the births in the first n generations.

Given $\mathfrak{F}^n$, the point process formed by the children of an n-th generation person at X has the same distributions as the process with points $\{Z_r^{(1)} + X\}$.

Let S be the event that there are people in every generation.

Let $I_{r_n}^{(n)} = \frac{Z_{r_n}^{(n)}}{n}$ and for each n, $\mathscr{P}^{(n)}$ be the set of points $\{I_{r_n}^{(n)}\}$.

$\mathscr{H}^{(n)}$: the convex hull of $\mathscr{P}^{(n)}$

If $x, y \in \mathbb{R}^n$, $\langle x, y\rangle$ is their inner product and $\|x\|$ is the Euclidean norm of x. The unit sphere is $S^{n-1} = \{X : \|x\| = 1\}$ and the closed ball of radius r, $B_r = \{x : \|x\| \leq r\}$.

The function $k(\theta)$ on $\mathbb{R}^n$ is defined by

$$k(\theta) = \log E\left[\sum_r \exp\langle -\theta, Z_r^{(1)}\rangle\right]$$

$k(0)$ is a number of initial ancestoers and assume that $k(0) > 0$.

Let the measure g be defined by $g(D) = E[\sharp\{r : Z_r^{(1)} \in \mathscr{D}\}]$ where $D \subset \mathbb{R}^n$ then

$$\exp k(\theta) = E\left[\sum_r e^{-\langle \theta, Z_r^{(1)} \rangle}\right] = \int \exp\langle -\theta, X\rangle dg(X)$$

This is a Laplace stileltjes transform of g.(Note that $\mathscr{L}(f)(s) = \int_0^\infty f(t)e^{-st}dt$). Also $k(\theta)$ is a convex function and let B is a convex set(it is possibly empty.) such that $k(\theta)$ is finite.
We will consider when B is not empty and $0 \in intB$. B is empty when the number of first generation people are infinite and $0 \notin intB$ when there exists $Z_r^{(1)}$ such that its norm is infinite.
The function ξ on $\mathbb{R}^n$ is given by

$$\xi(y) = \inf\{k(\theta) + \langle \theta, y \rangle : \theta\}$$

Let $\mathscr{P}(a)$ be

$$\mathscr{P}(a) = \{y : \xi(y) \geq a\}$$

and let $\mathscr{P}(0) = \mathscr{P}$

2 Multivariate Laplace-Stieltjes transforms

Lemma 2.1. *(i) $\mathscr{P}(a)$ is a closed convex set and $\mathscr{P}(a) = \cap_{d<a}\mathscr{P}(d)$.*
(ii) If $a < k(0)$ then $\mathscr{P}(a)$ is non-empty and $int\mathscr{P}(a) \subset \cup_{d>a}\mathscr{P}(d)$.
(iii) if $a < k(0)$ then $int\mathscr{P}(a)$ is non-empty if and only if $intA$ is non-empty.

Lemma 2.2. *If $0 \in intB$ then $\mathscr{P}(a)$ is compact.*

3 The shape of $\mathscr{H}^{(n)}$

Theorem 3.1. *For any $a < 0$, $\mathscr{P}^{(n)} \subset \mathscr{P}(a)$ for all but finitely many n on S.*

Theorem 3.2.

$$int\mathscr{P} \subset \liminf \mathscr{H}^{(n)} \subset \limsup \mathscr{H}^{(n)} \subset \mathscr{P} \qquad a.s. \text{ on } S$$

where $\liminf \mathscr{H}^{(n)} = \cup_{m\geq 1} \cap_{n>m} \mathscr{H}^{(m)}$ *and* $\limsup \mathscr{H}^{(n)} = \cap_{n\geq 1} \cup_{m\geq n} \mathscr{H}^{(m)}$.

Example 3.3. Let's think of branching random walk in $\mathbb{R}^1$. It starts with one initial ancestor at the origin and $\{Z_1^{(1)}\} \subset \{-1, 0, 1\}$ with $P(-1 \in \{Z_1^{(1)}\}) = P(1 \in \{Z_1^{(1)}\}) = p$ and $P(0 \in \{Z_1^{(1)}\}) = 1$. Then

$$k(\theta) = \log(e^{-\theta}p + e^{\theta}p + 1)$$

y is the right most point of $\mathscr{P}$, when y is minimum(or infimum) of $k(\theta)/\theta$ where $\theta < 0$. We can draw a graph of it. When $p = 1$, $y = 1$ so that $\mathscr{P} = [-1, 1]$. When $p = 0$, $y = 0$ and $\mathscr{P} = \{0\}$.

Proof. (Proof of theorem 3.1) Let's assume B is not empty. For any $h : \mathbb{R}^n \to \mathbb{R}^+$,

$$E\Big[\sum_r h(Z_r^{(n)})|\mathfrak{F}^{n-1}\Big] = \sum_r \int h(Z_r^{(1)} + X)dg(X)$$

In particular

$$E\Big[\sum_{r_n} \exp\langle -\theta, Z_{r_n}^{(n)}\rangle|\mathfrak{F}^{n-1}\Big] = \sum_{r_{n-1}} \int \exp\langle -\theta, Z_{r_{n-1}}^{(n-1)} + X\rangle dg(X) = \exp k(\theta) \sum_{r_{n-1}} \exp\langle -\theta, Z_{r_{n-1}}^{(n-1)}\rangle$$

So,

$$E\Big[\sum_r \exp\langle -\theta, Z_r^{n}\rangle\Big] = \exp nk(\theta)$$

Hence, when $\theta \in B$,

$$E\Big[\sum_{r_n} \frac{1}{\exp n(k(\theta) + \langle \theta, I_{r_n}^{(n)} \rangle)}\Big] = 1 \qquad \text{for each } n$$

Let Ω_n be the event that $\mathscr{P}^{(n)} \backslash \mathscr{P}(a)$ is non-empty, where $a < 0$ is fixed. Take $I_i^{(n)} \in \mathscr{P}^{(n)} \backslash \mathscr{P}(a)$ when Ω_n occurs.

By the definition of $\xi(y)$, if $\xi(y) < \infty$, there exists θ such that $\xi(y) \leq k(\theta) + \langle \theta, y \rangle \leq \xi(y) + \ln \epsilon$ where $\epsilon > 1$ and $e^a \epsilon < 1$. Since $I_i^{(n)} \notin \mathscr{P}(a)$,

$$\frac{1}{\exp(k(\theta) + \langle \theta, I_i^{(n)} \rangle)} \geq \frac{1}{\epsilon \exp \xi(I_i^{(n)})} \geq \frac{1}{\epsilon e^a}$$

For each $I_i^{(n)}$, choose corresponding θ_i such that $\xi(I_i^{(n)}) \leq k(\theta) + \langle \theta, I_i^{(n)} \rangle \leq \xi(I_i^{(n)}) + \ln \epsilon$. Now we have $I_i^{(n)}$ and θ_j such that $\sharp\{i\} = \sharp\{j\}$ and when $i = j$, above relation holds.

$$\begin{aligned}
\sharp\{i\} = \sharp\{j\} = \sum_i E\Big[\sum_j \frac{1}{(\exp(k(\theta_i)) + \langle \theta_i, I_j^{(n)} \rangle)^n}\Big] \\
\geq E\Big[\sum_i \frac{1}{(\exp(k(\theta_i)) + \langle \theta_i, I_i^{(n)} \rangle)^n}\Big] \\
\geq P(\Omega_n) E\Big[\sum_i \frac{1}{(\exp(k(\theta_i)) + \langle \theta_i, I_i^{(n)} \rangle)^n} \Big| \Omega_n\Big] \\
\geq P(\Omega_n) \sum_i (\epsilon e^a)^{-n} \geq P(\Omega_n) \sharp\{i\} (\epsilon e^a)^{-n}
\end{aligned}$$

Thus, we have

$$P(\Omega_n) \leq (\epsilon e^a)^n$$

Theorem 3.4. *(Borel Cantelli lemma) If $E_1, E_2, \cdots$ be a sequence of events in some probability space. If the sum of the probabilities of E_n is finite and $\sum_{n=1}^{\infty} P(E_n) < \infty$, then the probability that infinitely many of them occur is 0, i.e. $P(\limsup_{n \to \infty} E_n) = 0$*

By Borel Cantelli lemmma, $\mathscr{P}^{(n)} \subset \mathscr{P}(a)$ for all but finitely many n on S. □

Proof. (Proof of theorem 3.2) By lemma 2.1, $\mathscr{P}(a)$ is a closed convex set. For sufficiently large n, $\mathscr{P}^{(n)} \subset \mathscr{H}^{(n)} \subset \mathscr{P}(a)$ by theorem 3.1. For any $a < 0$, there exists N_a such that $\cup_{m>N_a}\mathscr{H}^{(m)} \subset \mathscr{P}(a)$. Since $\limsup \mathscr{H}^{(n)} \subset \cup_{m>N}\mathscr{H}^{(m)}$ for any N, $\limsup \mathscr{H}^{(n)} \subset \mathscr{P}(a)$ for any $a < 0$. By lemma 2.1, $\limsup \mathscr{H}^{(n)} \subset \mathscr{P}$.

Since $\liminf \mathscr{H}^{(n)} \subset \limsup \mathscr{H}^{(n)}$ is trivial, it suffices to show that $int\mathscr{P} \subset \liminf \mathscr{H}^{(n)}$. For this, we have to use 1 dimensional result.

Theorem 3.5. *Suppose $n = 1$, i.e. $\mathbb{R}$. and $k(\theta) < \infty$ for some $\theta > 0$. Let $\log(\mu(a)) = \inf\{\theta a + k(\theta) : \theta \geq 0\}$, $\gamma = \inf\{a : \mu(a) > 1\}$ and $I^{(n)}_{\min} = \inf\{I^{(n)}_r : r\}$. Then, $I^{(n)}_{\min} \to \gamma$ a.s. on S.*

The important obesrvation The projection of the branching random walk on $\mathbb{R}^n$ onto any subspace of $\mathbb{R}^n$ gives another branching random walk.

Let's suppose $0 \in intB$. By lemma 2.2, $\mathscr{P}$ is compact. Since $0 \in intB$, for any $y \in \mathbb{R}^n$, there exists $\theta > 0$ such that $\theta y \in S$ so that $k(\theta y) < \infty$. Let

$$\gamma(y) = \inf\{a : \inf\{k(\theta y) + \theta a : \theta \geq 0\} > 0$$

By theorem 3.5, there exists a sequence $\{I^{(n)}\}$ such that $\langle I^{(n)}, y\rangle \to \gamma(y)$ a.s. on S.

A point E in the convex set D is called an exposed point if there exists a supporting plane $\{x : \langle x, y\rangle = \kappa\}$ to D for which $D \cap \{x : \langle x, y\rangle = \kappa\} = E$.

Suppose that $\{x : \langle x, y\rangle = \kappa\}$ is a supporting plane to $\mathscr{P}$ such that $\mathscr{P} \subset \{x : \langle x, y\rangle \geq \kappa\}$. By lemma 2.1 (i), for any $\epsilon > 0$, $\mathscr{P}(a) \subset \{x : \langle y, x\rangle \geq \kappa - \epsilon\}$ for $a < 0$ sufficiently small. By theorem 3.1, $I^{(n)} \in \mathscr{P}(a) \subset \{x : \langle y, x\rangle \geq k - \epsilon\}$ for large n and $\langle I^{(n)}, y\rangle \to \gamma(y)$. Therefore, $\gamma(y) \geq \kappa - \epsilon$. Since ϵ is arbitrary $\gamma(y) \geq \kappa$.

Take $x \in int\mathscr{P}$. By lemma 3.(ii) $int\mathscr{P}(0) \subset \cup_{d>0}\mathscr{P}(d)$. $x \subset \mathscr{P}(d)$ for some positive d. Therefore, $\xi(x) > 0$. For all real θ, $k(\theta y) + \theta\langle y, x\rangle > 0$. By definition of $\gamma(y)$, $\gamma(y) \leq \langle y, x\rangle$. Since $\{x : \langle x, y\rangle = \kappa\}$ is a supporting plane to $\mathscr{P}$, we can choose x such that $\gamma(y) \leq \langle y, x\rangle \leq \kappa + \epsilon$. Thus, $\gamma(y) \leq \kappa + \epsilon$ and $\kappa = \gamma(y)$.

Any supporting plane to $\mathscr{P}$ has the form $\{z : \langle z, y\rangle = \gamma(y)\}$ for some $y \in S$. For any exposed point E of $\mathscr{P}$, $\exists y_0 \in B$ such that $\mathscr{P} \cap \{z : \langle z, y_0\rangle = \gamma(y_0)\} = E$.

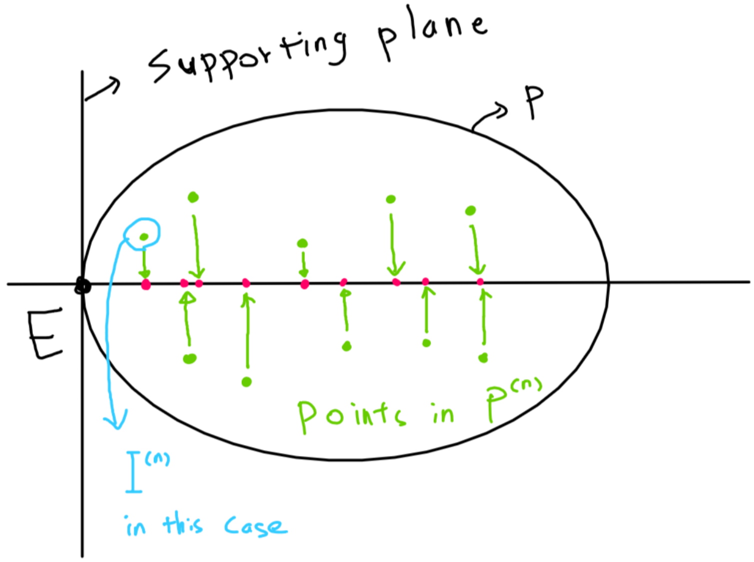

Figure 1

Now we take $\{I^{(n)}\}$ satisfying theorem 3.5 with $y = y_0$, then by theorem 3.1, $\{I^{(n)}\} \subset \mathscr{P}^{(n)} \subset \mathscr{P}(a)$ for all but finitely many n and $\mathscr{P}(a)$ is bounded. Thus, $\{I^{(n)}\}$ is bounded. Any accumulation point of it must lie in $\mathscr{P}(a) \subset \mathscr{P}$.

Let z_0 be an accumulation point of the sequence. There is a subsequence of $\{I^{(n)}\}$ such that $\langle I_k^{(n)}, y_0\rangle \to \langle z_0, y_0\rangle = \gamma(y_0)$. Thus, accumulation point lies in $\mathscr{P}$ and $\{z : \langle z, y_0\rangle = \gamma(y_0)\}$. It means $z_0 = E$ and the whole sequence must converge to E because there is only one accumulation point in a sequence $\{\langle I^{(n)}, y_0\rangle\}$. Therefore, $\left\|I^{(n)} - E\right\| \to 0$ as $n \to \infty$ a.s. on S.

Let $E_1, \cdots, E_N$ be exposed points of $\mathscr{P}$ and let $\mathscr{H}(E_1, \cdots, E_N)$ be their convex hull. We can choose n_i such that $\left\|E_i - I_i^{(k)}\right\| \leq \epsilon$ when $k \geq n_i$. Thus,

$$
\begin{aligned}
int\mathscr{H}(E_1, \cdots, E_N) &\subset int\mathscr{H}(I_1^{n_i}, \cdots, I_N^{n_{N_i}}) + B_\epsilon \subset \cap_{n \geq \max\{n_1, \cdots, n_N\}} \mathscr{H}^{(n)} + B_\epsilon \\
&\subset \liminf \mathscr{H}^{(n)} + B_\epsilon
\end{aligned}
$$

Thus, $int\mathscr{H}(E_1, \cdots, E_N) \subset \liminf \mathscr{H}^{(n)}$ a.s. on S.

As N increases, $int\mathscr{H}(E_1, E_2, \cdots E_N)$ approximates to $int\mathscr{P}$.

$$int\mathscr{P} \subset \liminf \mathscr{H}^{(n)} \qquad \text{a.s. on } S$$

□

References

[1] J.D.Biggins *The asymptotic shape of the branching random walk, Adv. Appl prob, (1978), 62-84*